Modern Physics

Problem Sheet 2: Wave functions

October 19, 2016

NB: For the consistent set of units to be used in these exercise please see the document "Physical Units" in the course webpage.

Exercise 1: An Eigenstate of the Quantum Harmonic Oscillator

Consider a quantum particle of mass m confined to move on a line (1-dimension) and interacting with a harmonic potential $V(x) = \frac{1}{2}m\omega^2 x^2$.

- Write the time-dependent Schrödinger equation describing the time-evolution of the wave-function of the particle
- Show that the corresponding stationary Schrödinger equation reads

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{1}{2}m\omega^2 x^2\psi(x) = E\psi(x) \tag{1}$$

- Consider an ansatz for a specific solution (eigenstate), in the form:

$$\psi(x) = e^{-\alpha x^2} \tag{2}$$

  Determine α for which this equation is satisfied. Hint: find the value of α such that the left-hand side of equation (1) reduces to the form

$$\text{const.} \times \psi(x)$$

  (as it should in order to match the right hand side).

- What is the corresponding energy (i.e. the eigenvalue of $\hat{H}$)?
- Compute the normalisation of this wave function, which ensures $\int_{-\infty}^{\infty} dx|\psi(x)|^2 = 1$.

Exercise 2: General solution

All eigenstates and eigenvalues of the quantum harmonic oscillator have been computed exactly. The normalised eigenstates (wave-functions) are given by the formula

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\ \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\ e^{-\frac{m\omega^2 x^2}{2\hbar}}\ H_n(x) \tag{3}$$

where $H_n(x)$ denote the so-called Hermite polynomial of order n and reads:

$$H_n(z) = (-1)^n e^{z^2}\frac{d^n}{dz^n}\ (\ e^{-z^2}\ ) \tag{4}$$

the corresponding energies (i.e. eigenvalues of $\hat{H}$) are given by

$$E_n = \hbar\omega\ \left(n + \frac{1}{2}\right) \tag{5}$$

- Check that the solution found in the previous exercise corresponds to $\psi_0(x)$ with energy E_0.
- Use the formulas (3) and (4) to compute the explicit form of the first excited state $\psi_1(x)$.
- Check that the corresponding eigenvalue E_1 matches the result of eq. (5) by substituting $\psi_1(x)$ into equation (1).