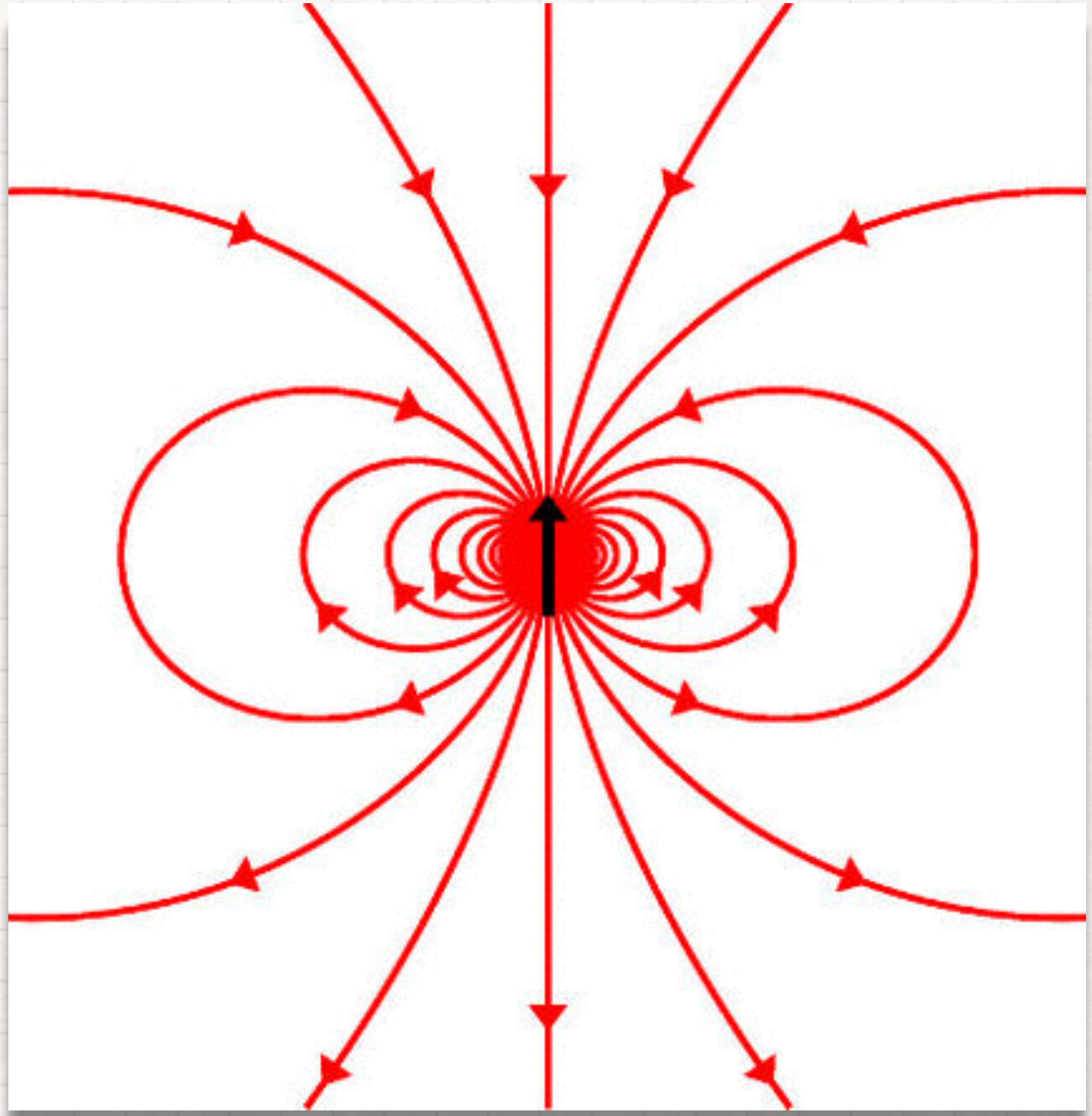
CHAPTER 5

ANGULAR MOMENTUM IN QUANTUM MECHANICS



Orbital Angular Momentum

Following the quantisation rules defined in the previous chapter we obtain an expression for the orbital angular momentum in position representation:

$$\hat{\vec{L}} = \left(-i\hbar\hat{\vec{\nabla}}\right) \times \hat{\vec{r}}$$

Direct inspection reveals (see problem sheet):

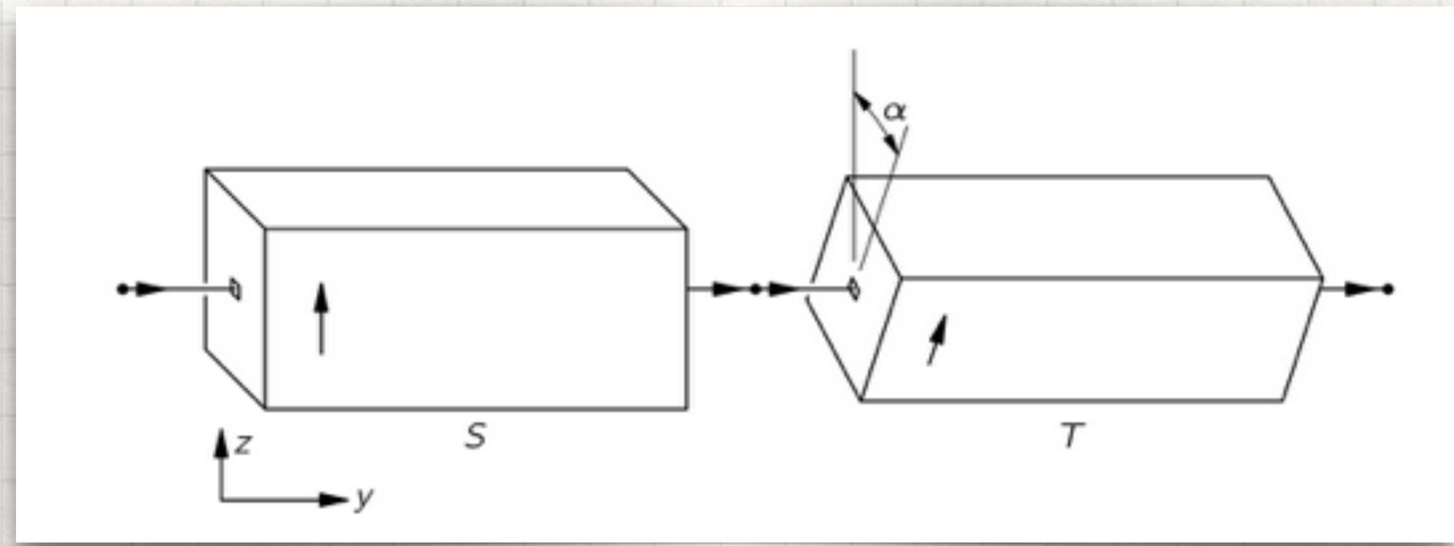
$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$$
$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$$
$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$$

it is impossible to measure with infinite accuracy more than one component (Uncertainty Principle for angular momentum)

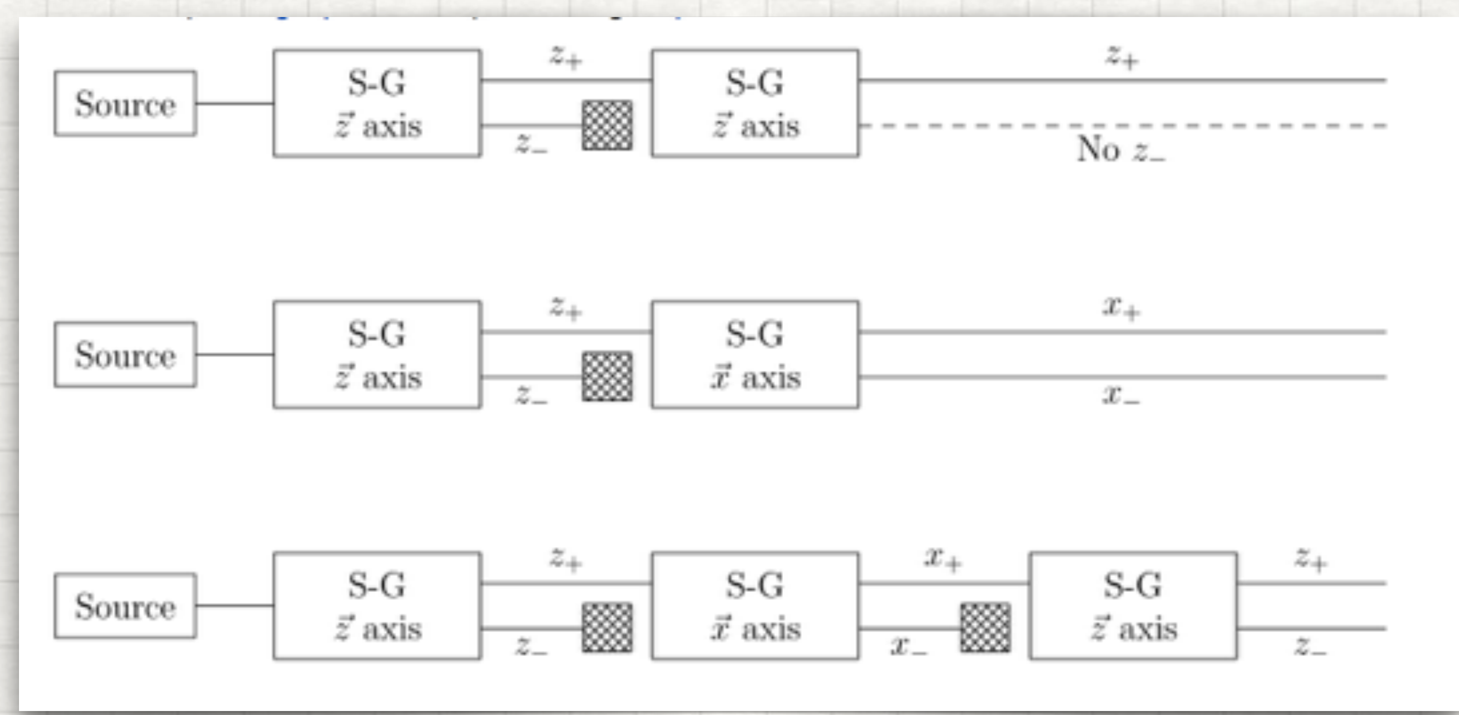
Furthermore: $[\hat{L}_x, \hat{L}^2] = [\hat{L}_y, \hat{L}^2] = [\hat{L}_z, \hat{L}^2] = 0$

Thus **it is possible** to simultaneously determine the magnitude and one component of **L**

Recall also:



Information on Lz destroys information on Lx, Ly!



Uncertainty Principle

It is only possible to specify the modulus of the angular momentum and the value of its projection along ONE axis.

From commutation relationships we can find states which are eigenstates to L^2 and L_z:

we denote them by $|l, m\rangle$

eigenvalue of L^2 (l)

eigenvalue of L_z (m)

Spectrum of L^2:

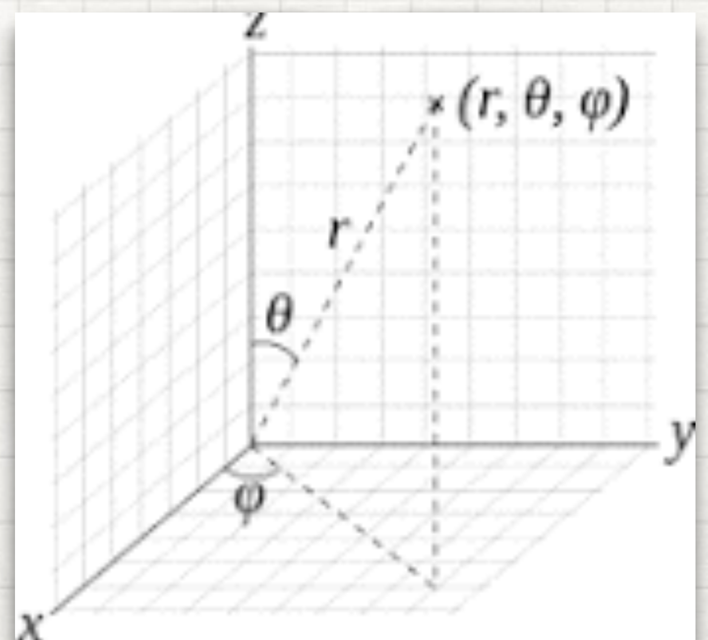
$$\hat{L}^2|l, m\rangle = \hbar^2 l(l+1)|l, m\rangle \qquad l = 0, 1, 2, \ldots$$

Spectrum of L_z:

$$\hat{L}_z|l, m\rangle = \hbar m|l, m\rangle \qquad m = -l, -l+1, .., 0, .., l-1, l$$

Advanced topic:
Eigenstates and spectrum of Angular Momentum

To derive the spectrum and to construct the related eigenfunctions it is useful to introduce spherical coordinates:



$$
\begin{aligned}
x &= r \sin\theta \cos\varphi \\
y &= r \sin\theta \sin\varphi \\
z &= r \cos\theta
\end{aligned}
$$

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \arccos \frac{z}{r} \\
\varphi &= \arctan \frac{y}{x}
\end{aligned}
$$

All differential operators can be expressed in spherical coords. In particular, the Laplacian reads:

$$
\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right)
$$

The quantum angular momentum operator is

$$\hat{L} = (-i\hbar\nabla) \times \hat{r}$$

In polar coordinates one finds

$$\hat{L}^2 \quad = \quad -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

We need to find functions such that

$$\hat{L}_z Y_{lm}(\theta,\phi) = \hbar^2 l(l+1) Y_{lm}(\theta,\phi)$$
$$\hat{L}^2 Y_{lm}(\theta,\phi) = \hbar m Y_{lm}(\theta,\phi)$$

These functions have been found in mathematical physics:

$$Y_\ell^m(\theta,\varphi) = (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\varphi}$$

(SPHERICAL HARMONICS)

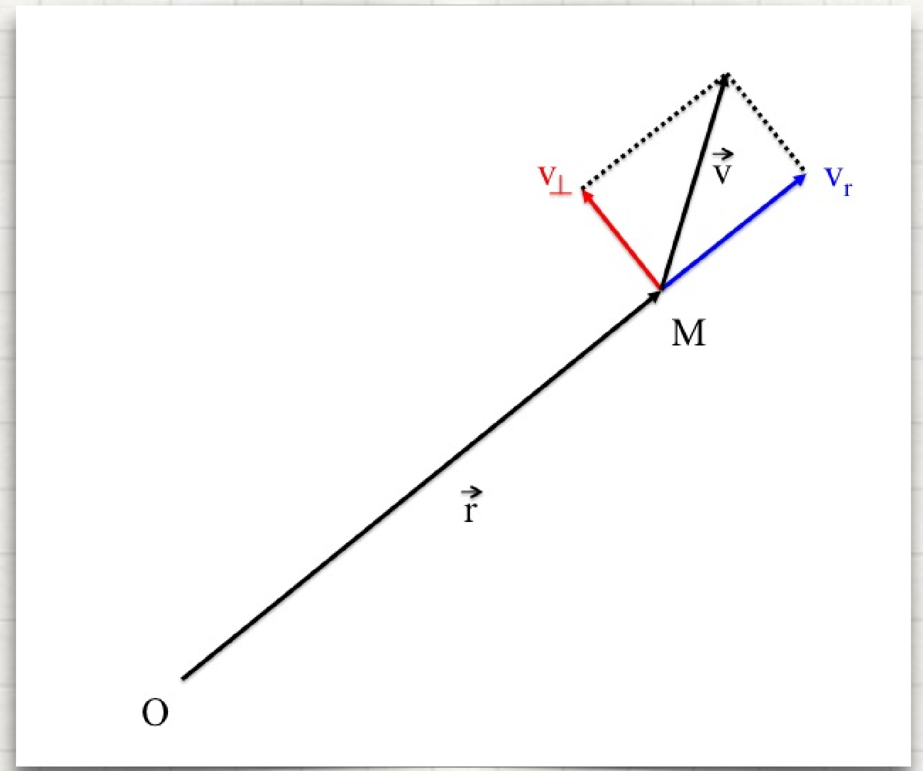
$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\,\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell.$$

(Associated Laguerre polynomials)

Advanced topic:

Motion of a quantum particle in a central potential

First review the classical discussion:



$$v_r = \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$|\vec{r} \times \vec{v}| = r|\vec{v}_\perp|$$

$$|\vec{\mathcal{L}}| = |\vec{r} \times \mu\vec{v}| = \mu r|\vec{v}_\perp|$$

If the potential depends only on r=|**r**|, then

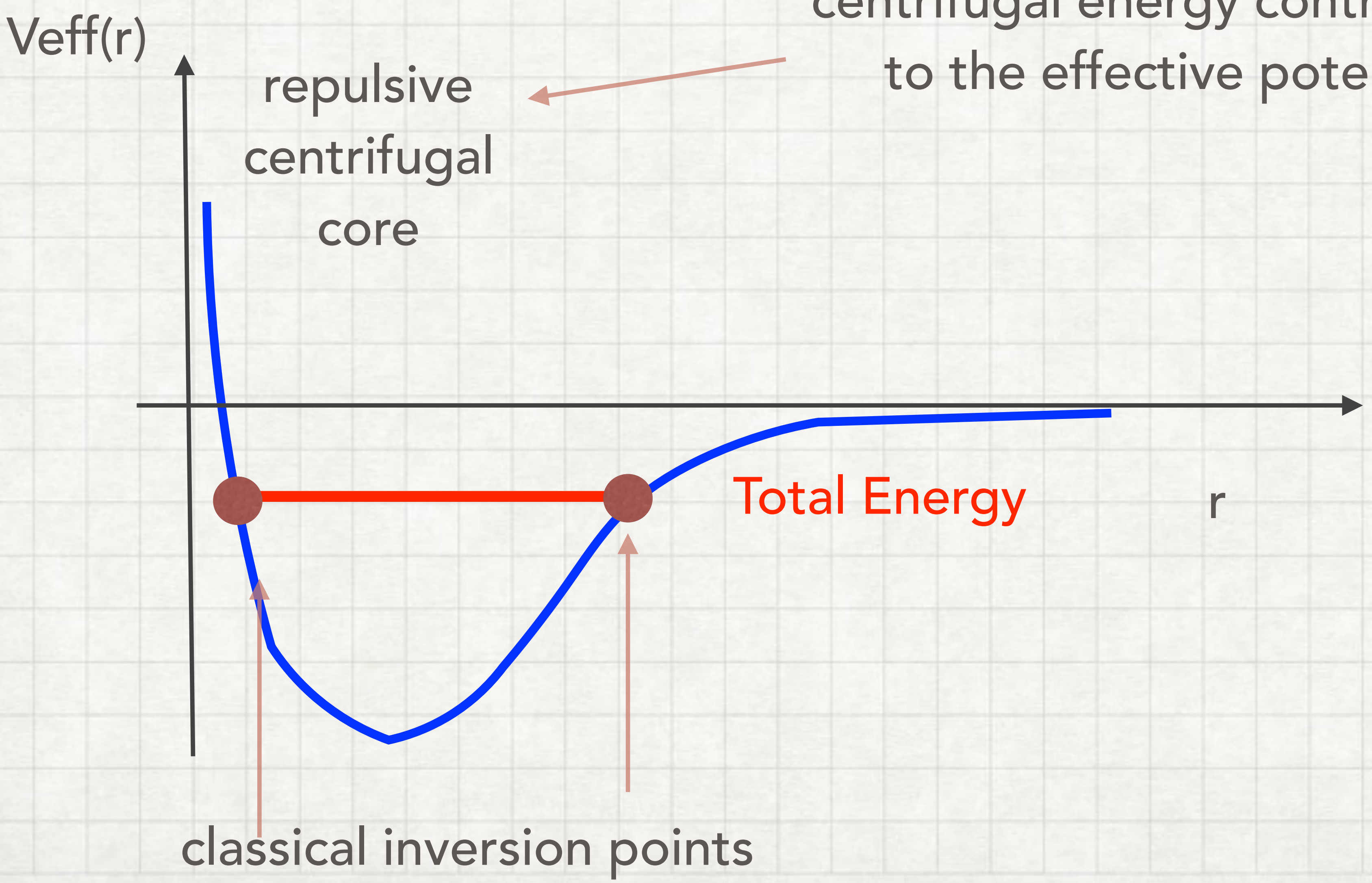
$$\frac{d}{dt}\mathbf{L}(t) = 0$$

The total energy of a particle in a central potential

$$\mathbf{H} = \frac{1}{2}\mu\vec{v}^2 + V(r) = \frac{1}{2}\mu\vec{v}_r^2 + \frac{1}{2}\mu\vec{v}_\perp^2 + V(r) = \frac{1}{2}\mu v_r^2 + \frac{\vec{\mathcal{L}}^2}{2\mu r^2} + V(r)$$

$$H = \frac{1}{2} m \dot{r}^2 + V(r) + \frac{\mathcal{L}^2}{2mr^2} \equiv \frac{1}{2} m \dot{r}^2 + V_{eff}(r)$$

centrifugal energy contribution
to the effective potential



Quantum Mechanical treatment:

We need to solve the stationary Schrödinger equation:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\phi(r,\theta,\phi) = E\phi(r,\theta,\phi)$$

Recalling the expression for the Laplacian in polar coords:

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

and that: $\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$

we find:

$$\left[-\frac{\hbar^2}{2mr}\frac{\partial^2}{\partial r^2}r + V(r) + \frac{\hat{L}^2}{2mr^2}\right]\phi(r,\theta,\phi) = E\phi(r,\theta,\phi)$$

We now make the following ansatz

$$\phi(r,\theta,\phi) = R(r)\ Y_l^m(\theta,\phi)$$

We obtain a new equation for each different values of l !!!:

$$\left[-\frac{\hbar^2}{2mr}\frac{d^2}{dr^2}r + V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}\right] R(r) = ER(r)$$

Finally defining $R(r) \equiv \frac{1}{r}u(r)$

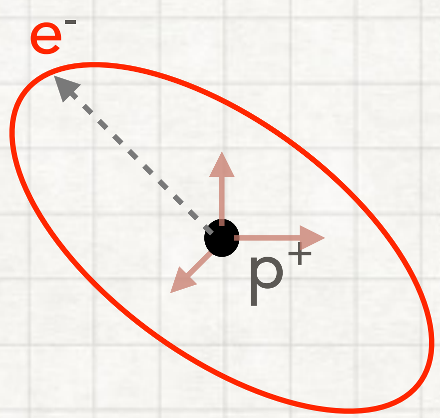
We obtain an equation which is formally analog to 1D Schrödinger equation but with an effective potential:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}\right] u(r) = Eu(r)$$

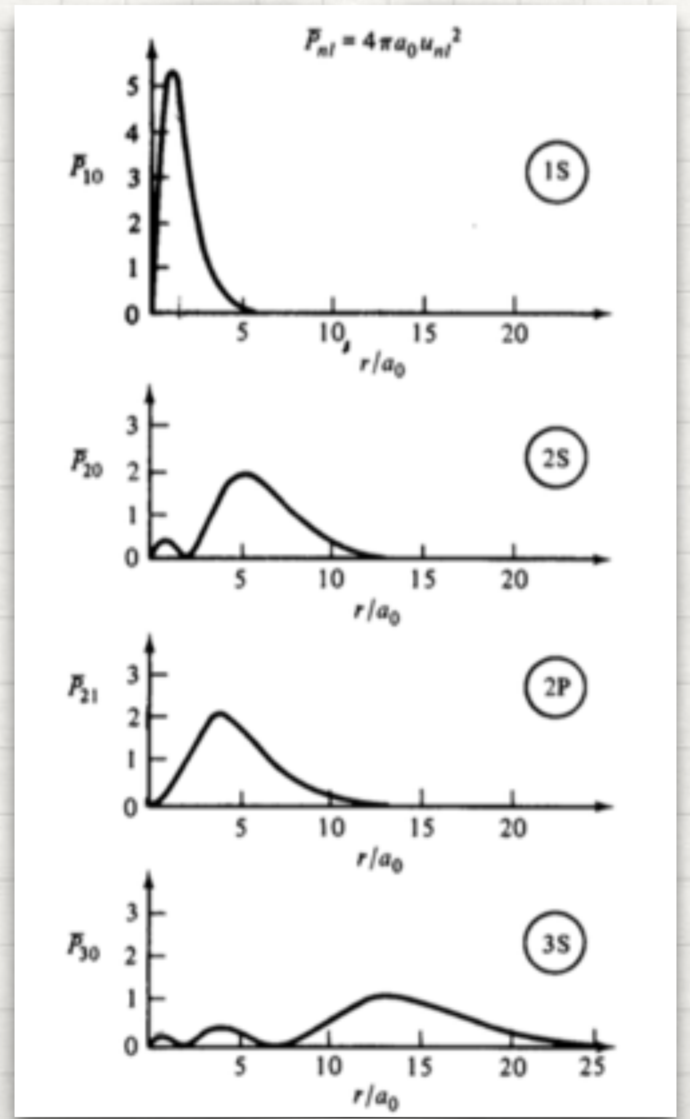
l=0 -> s-wave
l=1 -> p-wave
l=2 -> d-wave

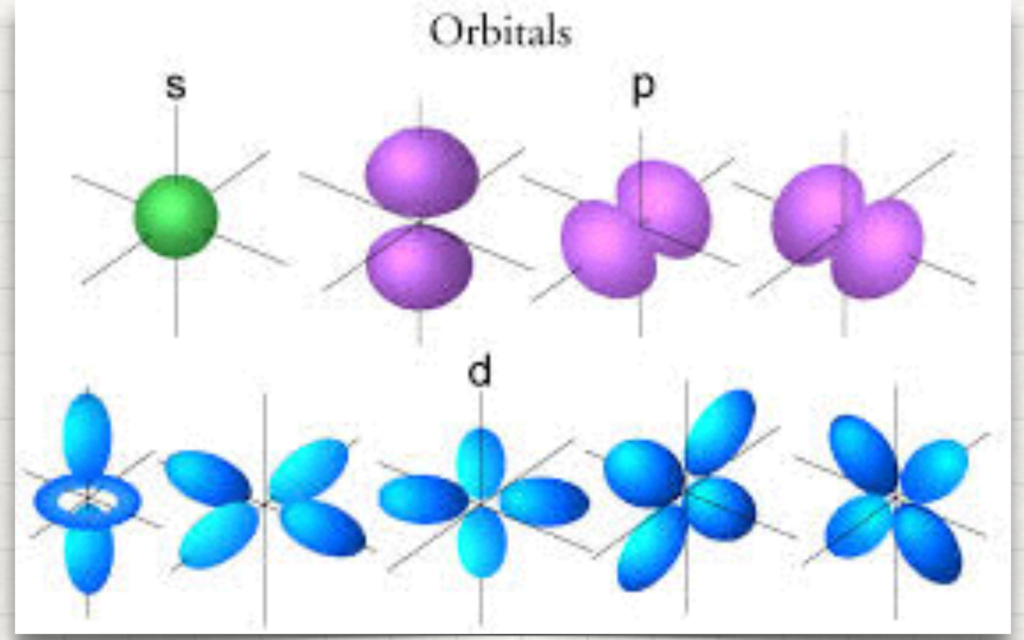
Complete analogy with classical case!

Electron's probability density in a hydrogen atom



$M_p >> m_e$: $\hat{H} = -\hbar^2 \frac{\nabla^2}{2m_e} - \frac{e^2}{r}$





Counter-intuitive predictions:
In some excited states, the electron cannot be found at certain
distances from the proton

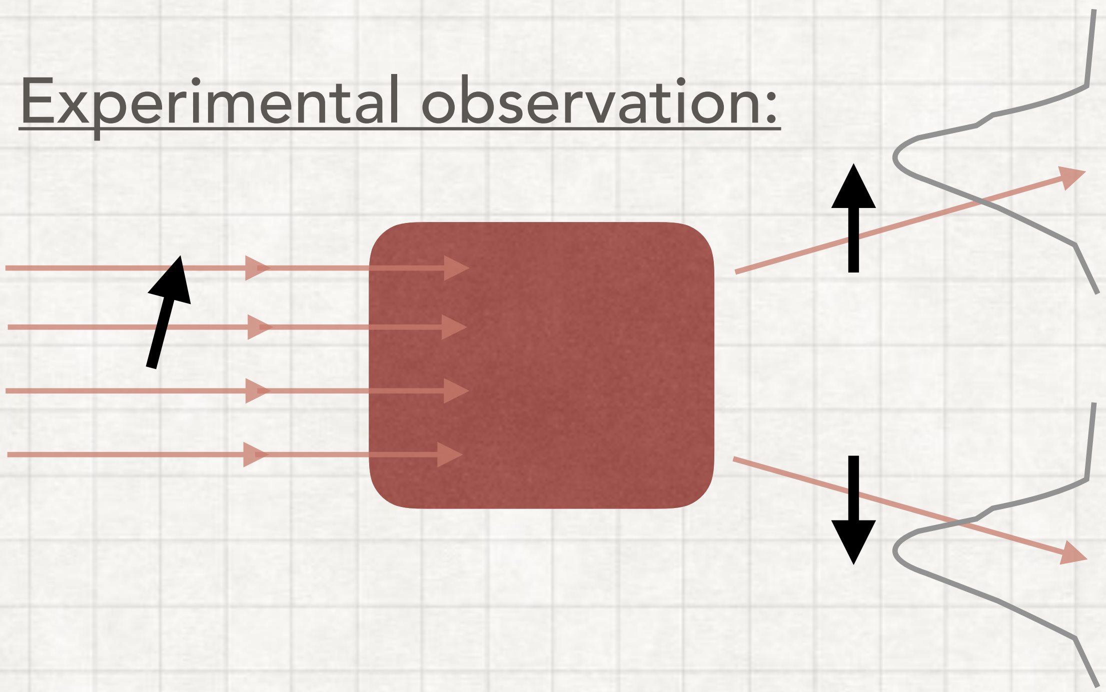
Angular momentum quantization

In electromagnetism, angular momentum is associated to a MAGNETIC MOMENT:

$$\vec{\mu} = \mu\vec{L}$$

Remember the Stern-Gerlach experiment:

<u>Experimental observation:</u>



All-up or all-down!

Magnetic moment (hence ang. mom.) IS QUANTIZED in integer or half-integer units of $\hbar$

In particular the splitting works as follows:

units of $\hbar$

0 ....no angular momentum....

L=1/2

L_z= +1/2 $\hbar$

L_z= -1/2 $\hbar$

L= 3/2

L_z= +3/2 $\hbar$

L_z= +1/2 $\hbar$

L_z= -1/2 $\hbar$

L_z= -3/2 $\hbar$

L=1

L_z= +1 $\hbar$

L_z= 0

L_z= -1 $\hbar$

L=2

L_z= +2 $\hbar$

L_z= +1 $\hbar$

L_z= 0

L_z= -1 $\hbar$

L_z= -2 $\hbar$

And so on.....

NB: L_z is a **quantum number**

Electrons protons and neutrons have spin 1/2, i.e.

$$\hat{S}^2|e\rangle = \hbar^2 \frac{1}{2}\left(1+\frac{1}{2}\right)|e\rangle$$

$$\hat{S}^2|p\rangle = \hbar^2 \frac{1}{2}\left(1+\frac{1}{2}\right)|p\rangle$$

$$\hat{S}^2|n\rangle = \hbar^2 \frac{1}{2}\left(1+\frac{1}{2}\right)|n\rangle$$

Non-relativistic spin 1/2 states are conveniently described by 2-dimensional vectors, called SPINORS

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli Matrixes

SPIN-STATISTICS CONNECTION

particles with half-integer spin are called **FERMIONS**:

(eg: electron, proton, neutron, neutrinos have all spin 1/2)

particles with integer spin are called **BOSONS**:

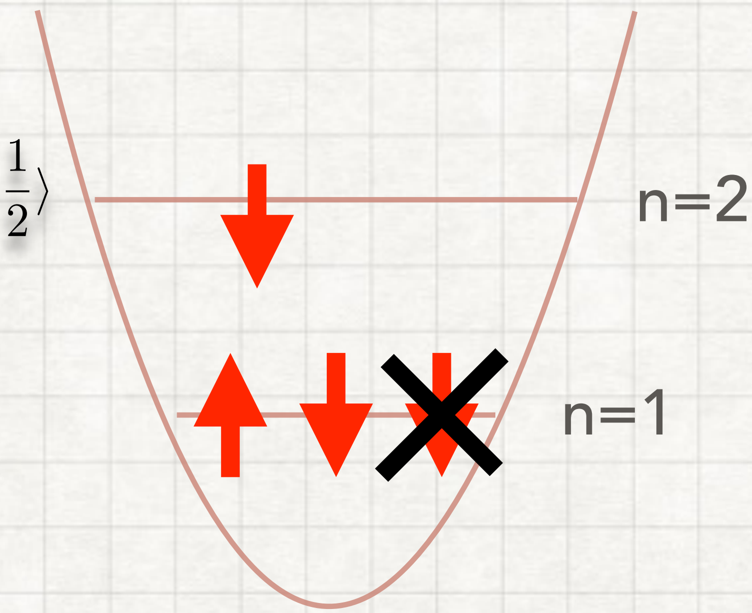
(eg: photon (S=1) pion (S=0), Higgs particle (S=0), …)

FERMIONS and BOSONS behave very differently in quantum mechanics! Indeed, their wave function is said to obey *different statistics.* Let's see what this is all about

PAULI EXCLUSION'S PRINCIPLE:

Two Fermions can not occupy the same quantum state

$|n=2, S_z=-\frac{1}{2}\rangle$



$|n=1, S_z=+\frac{1}{2}\rangle$ $|n=1, S_z=-\frac{1}{2}\rangle$

same collection of quantum numbers

$$|n, m, l, \ldots\rangle$$

DIRAC NOTATION

$$\langle n|\hat{O}|m\rangle = \int dx\ \phi_n^*(x)\hat{O}\phi_m(x)$$

SPIN-STATISTICS RELATIONSHIP

Pauli exclusion principle is automatically satisfied if one assumes that the wave function of identical BOSONS (FERMIONS) is symmetric (anti-symmetric) under exchange of particles

$$\psi(x_1, x_2) = \pm\psi(x_2, x_1)$$

if both particles are in the same point, $x_1 = x = x_2$ then:

$$\psi(x, x) = -\psi(x, x)$$ which can be true only if $$\psi(x, x) = 0$$

NB: Actually one does not need to assume the above relationship between spin e symmetry of the wave function. It is a theorem which follows from combining quantum mechanics with Einstein's special theory of relativity.